8. Elements of graph theory

To read:
[1] 8.1. How to Define Trees?,
[3] 4.1. The notion of a graph; isomorphism - only the definition of graphs, 4.3.1. Sum of the degrees, 4.3.2. Handshakes lemma, 5.1.

8.1. **Definition and characterizations of trees.**

Definition 8.1. A *graph* G is an ordered pair (V, E), where V is a set of elements called *vertices* and E is a set of 2-element subsets of V called *edges*.

Definition 8.2. Let $G = (V, E)$ be a graph. We call a sequence of distinct vertices $v_0, \ldots, v_r$ a *path* if $\{v_i, v_{i+1}\}$ is an edge of G, for every $0 \leq i \leq r - 1$.

Definition 8.3. We say that a graph $G = (V, E)$ is *connected* if for every two vertices $u, v \in V$ there exists a path in G between u and v.

Definition 8.4. For every vertex of a graph, we define its *degree* as the number of edges adjacent to it.

Definition 8.5. A *cycle* in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, \ldots, v_r \in V$ with $r \geq 3$ such that $v_r = v_0$ and $\{v_i, v_{i+1}\} \in E$ for all i from 0 to $r - 1$.

Definition 8.6. A *tree* is a connected graph without cycles.

Definition 8.7. A vertex of degree one in a tree is called a *leaf.*

Lemma 8.8. *Every tree on $n \geq 2$ vertices has at least two leaves.*

Proof. Let S be the set of all the paths in the tree T. We know that every path on r vertices contains exactly $r - 1$ edges. Consider now a path $v_1, \ldots, v_l$ of maximum length. One can always find a path of maximum length since every path in the tree can contain at most n vertices (otherwise it will be self-intersecting, that is it will contain a cycle, which is impossible since in a tree we cannot have cycles). We prove that both v_1 and v_l (the endpoints of the path) are leafs. Assume at least one of them is not, say v_1. That means that, there is at least another edge apart from $\{v_1, v_2\}$ incident to v_1. Observe that u cannot coincide with any of the vertices of the path $v_1, \ldots, v_l$ (otherwise it will close a cycle). Therefore, we can add u to the path without forming any cycle. But this is a contradiction to the maximality of the length of the path $v_1, \ldots, v_l$. Thus, both v_1 and v_l must be leaves. □

Theorem 8.9. *Every tree on n vertices has exactly $n - 1$ edges.*